ORDER OF OPERATIONS:

The Myth and the Math

Six thought-provoking issues challenge misconceptions about this iconic topic.

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Many of us embrace the order and beauty in mathematics. The order of operations is an iconic mathematics topic that seems untouchable by time, reform, or mathematical discoveries. Yet, think for a moment about a commonly heard statement in teaching the order of operations: “You work from left to right.” At another point in the curriculum, when working on properties of the operations, we say, “You can add numbers in any order” (commutative property). How can both of these statements be true? Preparing students to *do mathematics* means that they have an integrated understanding of rules and properties in mathematics.

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Rather than dismiss the order of operations as a convention established long ago, ***engage students in exploring equivalence*** *and see why operations are ordered as they are.*

The Common Core State Standards for Mathematics (CCSSM) (CCSSI 22010) introduces the order of operations in grade 3 and applies it in all later grades. How can we teach this fundamental topic effectively? First, we must be sure we understand it well ourselves. Test yourself. As you read each of the six statements below, decide if it is myth or math before reading the narrative that follows.

1. The order of operations was arbitrarily designed long ago.

Response: Myth

It is actually not true that "a long time ago, people just decided on an order in which operations should be performed . . . and it has stuck ever since" (Math Forum 2015). Two aspects of this myth are worthy of attention: First, that there is a long-standing consensus on the order. In fact, the debate is less than 100 years old and seems to have been driven by the beginning of textbook use in the early 1900s (Vanderbeek 2007). Cajori, American mathematician and author of *A History of Mathematical Notations* (1928–29), writes, "If an arithmetical or algebraical term contains ÷ and ×, there is at present no agreement as to which sign shall be used first" (vol. 1, p. 274).

The second aspect of this myth is that it is an arbitrary order, a convention. Conventions are ways of operating that could have just as easily been decided differently. For example, we put positive real numbers on the right side of the number line, but we could have made the opposite choice with no logical dilemma. Following this logic would imply that we arbitrarily decided on the order of operations, but if we think in terms of quantities and the representations of those quantities, it turns out that the order has a mathematical basis. Let's look at an example that will help use see *why* multiplication precedes addition:

$$4 + 3 \times 5$$

Because multiplication is repeated addition, we can rewrite this expression with an equivalent expression:

$$4 + 5 + 5 + 5$$

We could add these numbers in various ways. We might add the numbers from left to right, or we could first add the three 5s (e.g., 3×5) and then add on the 4. Both ways preserve the equivalence of the expression. Conversely, adding $4 + 3$ first in the expression $4 + 3 \times 5$ changes the mathematical meaning of the expression and does not preserve the equivalence. The same is true of an expression such as 2×5^3. Expanded, this means $2 \times 5 \times 5 \times 5$. To multiply 2×5 and then cube it changes the value of the expression. Rather than dismiss the order of operations as a convention established long ago, engage students in exploring equivalence and see why operations are ordered as they are.

2. The order of operations is rigid.

Response: Myth

Take a moment to think about what the properties of the operations tell us. We can (sometimes) rearrange numbers (commutative properties of addition and multiplication); we can sometimes group numbers differently (associative properties of addition and multiplication); and we can alter the order in which operations are completed (distributive property of multiplication over addition). Teaching the order of operations as a rigid set of rules is mathematically misguided and misses the opportunity to consider when we can and cannot apply the properties of the operations and preserve equivalence. In fact, the CCSSM Progression for K–Grade 5 Operations and Algebraic Thinking argues thus:

> Parentheses are important in expressing the associative and especially the distributive properties. These properties are at the heart of Grades 3 to 5 because they are used in the Level 3 multiplication and division strategies, in multi-digit and decimal multiplication and division, and in all operations with fractions. (CCSSI 2011, p. 28)

Let's look at another example:

$$5^3 + 4 \times 16 + 24 \times 4$$

According to the order of operations, the first step is to simplify 5^3 (125), then multiply 4×16 (64), then multiply 24×4 (96), then go from left to right to solve the addition ($125 + 64 + 96 = 189 + 96 = 285$). Yet, there are many other ways to simplify this expression. First,

any of the parts that are eventually added can be done in any order. Second, if you recognize that the 4 is multiplied by both the 16 and the 24 (distributive property), you can instead solve 4×40. That makes the problem $125 + 160 = 285$. When we teach order of operations in a rigid way, students miss out on opportunities to look for efficient approaches, a critical component of procedural fluency. The first of the Common Core's Standards for Mathematical Practice (SMP 1) states that mathematically proficient students look for entry points to a problem's solution:

> They analyze givens, constraints, relationships, and goals. They . . . plan a solution pathway rather than simply jumping into a solution attempt. (CCSSI 2010, p. 6)

In this example, mathematically proficient students should first look at the problem holistically and decide how to most efficiently find the solution, applying what they know about the properties of the operations and the order of operations. Problems that incorporate a variety of operations offer the opportunity to look for constraints (what will change the result of the operations) and relationships (which numbers or operations might lead to a more efficient solution path).

3. The order of operations can be taught conceptually.

Response: Math

Our goal to help students become mathematically proficient requires that we try to make the connection among concepts, procedures, and facts (CCSSI 2010; NRC 2001). When it comes to order of operations, we may wonder, How can such a procedurally focused topic as the order of operations be taught in a way to develop mathematical proficiency? Let's look at another example and see how it was discussed in a sixth-grade multicultural, multilingual classroom.

$$8 + 3 \times 5 + 7$$

The teacher, Ms. G, has just read *Two of Everything* (Hong 1993). In this story, Mr. Haktak discovers a large magic pot that doubles everything that goes into it. Students had heard the story earlier in the year when they explored

FIGURE 1

In this discussion, students are making sense of why multiplication precedes addition, and they are thinking flexibly about the order in which they can combine the numbers.

Ms. G: How many coins?

Emile: Thirty coins.

Ms. G: Any other answers? [*None are offered*.] What order did you use to solve this problem?

Francesca: I did three stacks of five coins first, five plus five plus five, then added seven and eight.

Leila: I did the same but just multiplied three times five.

Ms. G: Who else did these stacks first? [*All hands go up*.] What is next?

Anh: I have eight, fifteen, and seven, so I get thirty.

Ms. G: [*Writing on the board*] 8 + 15 + 7 = 30. In what order did you add them?

Makena: I added eight and seven, fifteen. Fifteen and fifteen is thirty.

José: I just added them across. Eight and fifteen is twenty-three, and then seven more is thirty.

Ms. G: Did the order that we added make a difference? Roberta and Jose added them in a different order. Did it matter?

Neesa: Not when it is all addition; then you can rearrange and add differently.

Ms. G: So, can we add eight plus five first? Talk to your partners and be ready to justify why or why not. [*Two minutes pass*.] What do you think?

Lorena: No, you can't. It doesn't make sense with the stacks.

Jason: There aren't three coins. The three just tells how many stacks of five, so that [three times five] has to be done first.

Angie: The story is about three stacks of five, not eleven stacks of five. If you add first, then it would be different—it would change the situation.

Ms. G: I think we have a conjecture from this problem: We need to multiply before we add, but when it is all addition, we can add in any order.

algebra. Ms. G returned to the story to show the illustration of the numerous stacks of coins that the Haktaks had produced from the doubling pot. Ms. G explained that they were going to help the Haktaks count their coins. She wrote $8 + 3 \times 5 + 7$ on the board and said, "The Haktaks have one stack of eight coins, three stacks of five coins, and one stack of seven coins. Tell me how many coins the Haktaks have."

Students were permitted to grapple with the task, and a discussion of their solutions followed (see **fig. 1**). With further experience using the context of stacking coins, these students will continue to develop a strong understanding of the order in which they can apply operations. In

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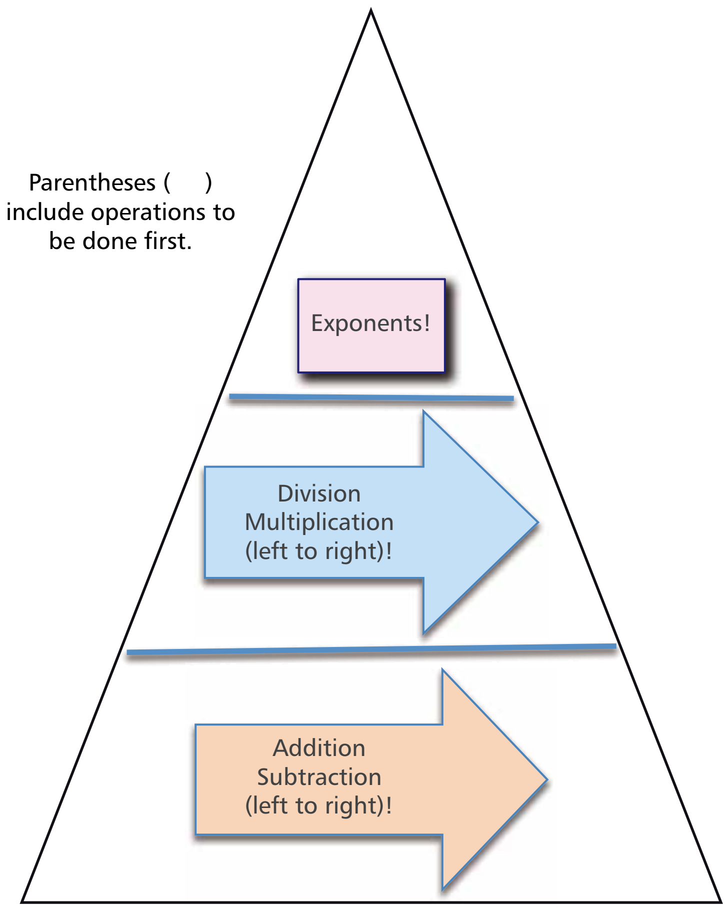
FIGURE 2

Below is an adaptation of the visual representation that Ameis (2011) suggests to illustrate that multiplication and division are at the same level in the hierarchy of operators.

Hierarchy of the order of operations
(adapted from Ameis 2011)

Operations higher in the hierarchy are completed first.



this way, they are looking for and making use of structure (SMP 8) (CCSSI 2010).

4. The order of operations is best taught using memory triggers.

Response: Myth

Preparing mathematically proficient students requires that what is learned is understood—every day, all topics. Otherwise, we send a confusing message to students that it is important to understand only *some* mathematics. Notice that statement 4 uses the word *taught*. Although the order of operations should be taught conceptually, as we saw in Ms. G's class, memory triggers can reinforce that instruction and can be particularly useful for students with disabilities. The two popular memory triggers in the United States (Please Excuse My Dear Aunt Sally and PEMDAS/PEDMAS) can help students remember and effectively apply the order of operations after it has been developed conceptually. Unfortunately, these triggers have caused major misconceptions about the order of operations. First, they imply that there are six steps in the order of operations. Second, students erroneously assume that multiplication precedes division and addition precedes subtraction (Jeon 2012). Consider this example:

$45 \div 5 \times 9$

If multiplication is done first, the answer will be 1, which is incorrect. The answer is 81. Visuals and other techniques can more accurately help students understand and remember the order of operations, as the next discussion will show.

5. Four operation steps are in the order of operations.

Response: Myth

Often the order is listed as (1) parenthesis, (2) exponents, (3) multiplication and division and (4) addition and subtraction. Parentheses are grouping symbols, not operation symbols. Therefore, there are only three operation steps. Ameis (2011) suggests the use of a triangle as a way to illustrate the hierarchy of operators (see **fig. 2**). Unlike the memory triggers, this visual prompt illustrates that multiplication and division are at the same level in the order of operations.

Because elementary school students are working only on the lower two levels, and with parenthesis, the top tier can be left blank or shaded so students can see that they will soon add another operation (exponents) in middle school (grade 6 in CCSSM).

6. The order of operations is universal.

Response: Math and myth

The basic order of operations (e.g., exponents first) is common across countries, but differences do exist within the tiers and with how they are described. Kenyans, for example, explain that division comes *before* multiplication (Maina 2012). Let's look at another example:

$100 \times 20 \div 5$

Applying the order of operations as it is described in the United States, this expression would be simplified from left to right: $100 \times 20 = 2000$ then divide $2000 \div 5 = 400$. In Kenya, students are taught to divide first: $20 \div 5 = 4$, then multiply: $100 \times 4 = 400$. It works! (Try more examples to convince yourself.) Consider the two options for this step: (1) multiply and divide in order from left to right (United States) and (2) divide before you multiply (Kenya). Either of these statements accurately describes the same step in the order of operations. Understanding the

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Students need the opportunity to explore the structure of numbers through meaningful tasks.

properties of multiplication and the relationship between multiplication and division is a focus in grade 3 CCSSM Operations and Algebraic Thinking. The two different conventions for doing multiplication and division are an opportunity to explore these properties and relationships. (Notice that the example could have been rewritten as $100 \times 5 \times 1/5$, and then the commutative property or associative property could be applied to illustrate why the Kenyan approach works.) These two options for describing this step of the order of operations can launch an excellent cultural as well as mathematical investigation: Ask students to determine if these statements can both be true and why they think so (using coins as described above, for instance).

Terminology also varies in different regions of the world. For example, in Canada, the United Kingdom, and other English-speaking countries, people refer to the steps as Brackets, Order, Division, Multiplication, Addition, and Subtraction (with acronyms of BODMAS, BEDMAS, and BIDMAS, depending on whether the second step is named *Order*, *Exponents*, or *Indices*). In the United States, the term PEMDAS or PEDMAS is more common. These differences may simply be due to how different textbooks in different countries began to describe the order of operations.

Misleading descriptions

Order of operations is often described as arbitrary, a rigid series of steps, best taught through memorization, and universally learned in one way by all students. These descriptors are misleading; such instruction denies students the opportunities to explore equivalence and properties of the operations—to really explore the structure of numbers and "do mathematics." Simplifying expressions that involve various operations must be a worthwhile venture into mathematical reasoning and sense making. The progression for K–Grade 5 Operations and Algebraic Thinking states the following:

> In Grade 6, students will begin to view expressions not just as calculation recipes but as entities in their own right, which can be described in terms of their parts. For example, students see $8 \times (5 + 2)$ as the product of 8 with the sum $5 + 2$. In particular, students must use the conventions for order of operations to *interpret* expressions, not just to evaluate them. Viewing expressions as entities created from component parts is essential for seeing the structure of expressions in later grades and using structure to reason about expressions and functions. (CCSSI 2011, p. 34)

For this to happen, we must teach the order of operations through meaningful tasks that use contexts (e.g., stacking coins) and engage students in problems that are focused on finding equivalent expressions (like comparing the Kenya explanation to the U.S. explanation for the order of multiplication and division). This approach is much more likely to help students become flexible, accurate, and efficient in simplifying expressions—in other words, procedurally fluent.

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